arXiv:math/0512068v1 [math.ST] 2 Dec 2005

Minimizing Squared Vertical and Squared Horizontal Errors

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Abstract

The slope of the best fit line from minimizing the sum of both the squared vertical errors and the squared horizontal errors is shown to be the root of a fourth degree polynomial.

1 Introduction

With simple linear regression we have data $\{(x_1, Y_1|X = x_1), ..., (x_n, Y_n|X = x_n)\}$ and we minimize the sum of the squared vertical errors. The question posed here is "Can we effectively minimize both the sum of the squared vertical and squared horizontal errors?" For notational convenience, we assume that the data is *positively* correlated.

As an example, suppose we have paired data (X, Y) where we first fit a linear function $f(x) = y = \beta_0 + \beta_1 x$ to the data. For example, Y could be the grade point average GPA at graduation from a four year university for a student, and X is the corresponding SAT score before matriculation. Typically, Admissions Committees use such a least-squares model to measure the effectiveness of the SAT scores in the admission process.

Suppose now we want to preform an inverse prediction at the value y_0 to answer the question "What SAT score should an admissions candidate receive in order to have a predicted GPA of, say, 2.0." This is found from the inverse function $f^{-1}(y) = x = y/\beta_1 - \beta_0/\beta_1$.

2 Model

For inverse prediction, we will want both $f(x)$ and $f^{-1}(y)$ to "fit" the data, and we hope that the squared vertical and squared horizontal errors will both be small for the fitted line $h(x) = \widehat{\beta}_0 + \widehat{\beta}_1 x$ which has minimized both the squared vertical and squared horizontal errors. To that end, set

$$SSE = \gamma \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 + (1 - \gamma) \sum_{i=1}^{n} (x_i - y_i/\beta_1 + \beta_0/\beta_1)^2 \qquad (1)$$

where γ $(0 \leq \gamma \leq 1)$. The parameter γ allows for a weighting of the two components of SSE yielding the least square estimators for $f(x)$ as $\gamma \to 1$, and the least square estimators of $f^{-1}(y)$ as $\gamma \to 0$.

We compute

$$\frac{\partial}{\partial \beta_0} SSE = \frac{2\left(n\beta_0 - \sum_{i=1}^{n}(y_i - \beta_1 x_i)\right)\left(\gamma\beta_1^2 + 1 - \gamma\right)}{\beta_1^2} \tag{2}$$

with root

$$\widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1 \overline{x} \tag{3}$$

independent of γ, the same as in simple linear regression.

To find the slope $\hat{\beta}_1$, .we compute

$$\begin{aligned} \frac{\partial}{\partial \beta_1} SSE \quad &= \quad \gamma \sum_{i=1}^{n} \left(-2x_i y_i + 2\beta_0 x_i + 2\beta_1 x_i^2\right) \qquad (4) \\ &\quad +(1-\gamma)\left(\frac{-2n\beta_0^2}{\beta_1^3} + \sum_{i=1}^{n}\left(\frac{2x_i y_i}{\beta_1^2} - \frac{2\beta_0 x_i}{\beta_1^2} - \frac{2y_i^2}{\beta_1^3} + \frac{4\beta_0 y_i}{\beta_1^3}\right)\right). \end{aligned}$$

Set $S_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2$, $S_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2$ and $S_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$, and let $\rho = S_{xy}/\sqrt{S_{xx}S_{yy}}$ denote the correlation.

After some manipulation, the roots of Equation 4 are found by solving

$$\gamma\sqrt{\frac{S_{xx}}{S_{yy}}}\beta_1^4 - \gamma\rho\beta_1^3 + (1-\gamma)\rho\beta_1 - (1-\gamma)\sqrt{\frac{S_{yy}}{S_{xx}}} = 0. \tag{5}$$

The (positive) root of Equation 5 will be the slope of the line which has minimized the γ-weighted sum of the squared vertical and squared horizontal errors.

With $\gamma = 1.00$, the slope $\widehat{\beta}_1 = \rho\sqrt{S_{yy}/S_{xx}}$; with $\gamma = 0.00$, the slope $\widehat{\beta}_1 = (1/\rho)\sqrt{S_{yy}/S_{xx}}$; and in general,

$$\rho\sqrt{S_{yy}/S_{xx}} \leq \widehat{\beta}_1 \leq (1/\rho)\sqrt{S_{yy}/S_{xx}} \tag{6}$$

3 An Example

Suppose the data set is $\{(0,0),(0,0),(1,0),(1,1)\}$ with $\{\overline{x} = 1/2, \overline{y} = 1/4, S_{xx} = 1, S_{yy} = 3/4, \rho = \sqrt{3}/3 = 0.5774\}$. If we choose $\gamma = 0.9$, from Equation 5, $\widehat{\beta}_1 = 0.6612$; and from Equation 3, $\widehat{\beta}_0 = 1/4 - (1/2)\widehat{\beta}_1 = -.08060$. The bounds for $\widehat{\beta}_1$ are given in (6) and are $1/2 \leq \widehat{\beta}_1 \leq 3/2$.